ON A GENERALIZED MAXIMUM PRINCIPLE FOR A TRANSPORT-DIFFUSION MODEL WITH log-MODULATED FRACTIONAL DISSIPATION

HONGJIE DONG AND DONG LI

Abstract. We consider a transport-diffusion equation of the form $\partial_t \theta + v \cdot \nabla \theta + \nu \mathcal{A}\theta = 0$, where v is a given time-dependent vector field on $\mathbb{R}^d$. The operator $\mathcal{A}$ represents log-modulated fractional dissipation: $\mathcal{A} = \frac{|\nabla|^\gamma}{\log^\beta(\lambda+|\nabla|)}$ and the parameters $\nu \ge 0$, $\beta \ge 0$, $0 \le \gamma \le 2$, $\lambda > 1$. We introduce a novel nonlocal decomposition of the operator $\mathcal{A}$ in terms of a weighted integral of the usual fractional operators $|\nabla|^s$, $0 \le s \le \gamma$ plus a smooth remainder term which corresponds to an L^1 kernel. For a general vector field v (possibly non-divergence-free) we prove a generalized L^∞ maximum principle of the form $\|\theta(t)\|_\infty \le e^{Ct}\|\theta_0\|_\infty$ where the constant $C = C(\nu,\beta,\gamma) > 0$. In the case $\mathrm{div}(v) = 0$ the same inequality holds for $\|\theta(t)\|_p$ with $1 \le p \le \infty$. At the cost of an exponential factor, this extends a recent result of Hmidi [7] to the full regime $d \ge 1$, $0 \le \gamma \le 2$ and removes the incompressibility assumption in the L^∞ case.

1. Introduction

We consider the transport equation with log-modulated fractional dissipation of the form

$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta + \nu \mathcal{A}\theta = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ \mathcal{A}\theta := \frac{|\nabla|^\gamma}{\log^\beta(\lambda+|\nabla|)}\theta, \\ \theta(0,x) = \theta_0, \end{cases} \tag{1.1}$$

where $\nu \ge 0$, and $v = v(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a given vector field, possibly non-divergence-free. The basic unknown is the scalar function $\theta = \theta(t,x)$ which is usually termed "active scalar". The operator $\mathcal{A}$ is defined via the Fourier transform

$$\widehat{\mathcal{A}f}(\xi) = \frac{|\xi|^\gamma}{\log^\beta(\lambda + |\xi|)}\hat{f}(\xi), \qquad \xi \in \mathbb{R}^d,$$

where the parameters $0 \le \gamma \le 2$, $\beta \ge 0$, $\lambda > 1$. It is termed log-modulated fractional dissipation since it is the usual fractional Laplacian operator $|\nabla|^\gamma$ divided by a logarithm symbol. Along the way we will also consider a variant of the operator $\mathcal{A}$ which is denoted as $\mathcal{A}_1 = |\nabla|^\gamma \log^{-\beta}(\lambda - \Delta)$, i.e.

$$\widehat{\mathcal{A}_1 f}(\xi) = \frac{|\xi|^\gamma}{\log^\beta(\lambda + |\xi|^2)}\hat{f}(\xi), \qquad \xi \in \mathbb{R}^d.$$

The main objective of this paper is to prove some maximum principles for the operator $\mathcal{A}$ and $\mathcal{A}_1$ in Lebesgue spaces.

The transport-diffusion model (1.1) is a natural generalization of several linear and nonlinear fluid equations such as the two-dimensional surface quasi-geostrophic

equations, fractional Burgers equations, vortex patch models, and Boussinesq systems. See, for instance, the recent work [9, 4, 3, 10, 6, 5] and references therein. In these problems, the velocity v is typically related to the active scalar θ by a constitutive relation $v = \mathcal{T}(\theta)$ where $\mathcal{T}$ could be some singular integral operator or more generally a nonlocal operator. To obtain local and global wellposedness results for the nonlinear problems, an important first step is to get a priori L^p, $1 \le p \le \infty$ estimates of solutions. Specific to the linear problem (1.1), one needs to prove L^p bounds on the active scalar θ *independent of the size of* v. We refer to these types of results as L^p maximum principle estimates. In this respect, the two-dimensional dissipative surface quasi-geostrophic equations can be regarded as a (nonlinear) version of (1.1) and they correspond to the case $\beta = 0$ in the operator $\mathcal{A}$. A classical result is due to A. Córdoba and D. Córdoba [2] who proved the following

$$\|\theta(t)\|_p \le \|\theta(0)\|_p, \quad \forall\, t \ge 0,\, 1 \le p \le \infty.$$

In a recent article, Hmidi [7] initiated the study of (1.1) and obtained the following important maximum principle:

Theorem 1.1 (Hmidi [7])**.** *Let the dimension $d = 2, 3$ and let $\nu \ge 0$, $0 \le \gamma \le 1$, $\beta \ge 0$, $\lambda \ge e^{\frac{3+2\alpha}{\beta}}$. Assume the velocity v is divergence-free, i.e. $\nabla \cdot v = 0$. Then any smooth solution of* (1.1) *satisfies*

$$\|\theta(t)\|_p \le \|\theta(0)\|_p, \quad \forall\, t \ge 0,\, 1 \le p \le \infty.$$

To prove Theorem 1.1, Hmidi used the theory of C_0-semigroup of contractions on L^p $(1 < p < \infty)$ for the family of convolution kernels $(K_t)_{t \ge 0}$ defined by

$$\widehat{K_t}(\xi) = e^{-t\frac{|\xi|^\gamma}{\log^\beta(\lambda+|\xi|)}}.$$

The key step is to get the positivity of the kernel K_t. For this purpose, Hmidi used the Askey's criterion for characteristic functions [1]. The restrictions on the dimension d and the parameters (γ, β, λ) are mainly due to the use of this criterion. Hmidi conjectured that the maximum principle should hold for all dimensions $d \ge 1$ and the full range $0 \le \gamma \le 2$ and $\beta \ge 0$. The purpose of this paper is to give an affirmative answer to this question at the cost of a harmless exponential factor.

Theorem 1.2 (Generalized maximum principle, L^∞ case)**.** *Let $\nu \ge 0$, $d \ge 1$, $0 \le \gamma \le 2$ and $\beta \ge 0$, $\lambda > 1$. Assume $\theta = \theta(t, x)$ is a smooth solution of* (1.1) *which decays at spatial infinity, i.e., for any fixed $t \ge 0$,*

$$\lim_{|x| \to \infty} \theta(t, x) = 0, \tag{1.2}$$

Then we have

$$\|\theta(t)\|_\infty \le e^{Ct} \|\theta_0\|_\infty, \quad \forall\, t \ge 0, \tag{1.3}$$

where $C > 0$ is a constant depending only on $(\nu, d, \gamma, \beta, \lambda)$.

Remark 1.3. The same result holds if we replace the dissipation operator $\mathcal{A}$ by $\mathcal{A}_1$ in (1.1). As we shall see in Section 2, the proof for $\mathcal{A}_1$ case is actually simpler. The decay condition (1.2) is fairly weak as most smooth solutions to these type of fluid equations typically belong to the Sobolev space $C_t^0 H_x^s$ which can easily imply (1.2). We should also stress that we do not assume any divergence-free condition on v in Theorem 1.2. This can have applications for compressible fluid equations.

To prove Theorem 1.2, we shall use a completely new idea which avoids the use of Askey's criterion. Namely we introduce a novel nonlocal decomposition of the operator $\mathcal{A}$ (see Section 2 for more details) in terms of a weighted integral of the usual fractional operators $|\nabla|^s$, $0 \le s \le \gamma$ plus a smooth remainder term which corresponds to an L^1 kernel. Thanks to this new decomposition, we shall only need to appeal to the classic maximum principle for the fractional Laplacian operators. In a similar vein, one can even consider a weighted integral of a parameterized family of nonlocal operators each of which obeys a maximum principle. However we shall not pursue this generality here.

As was already mentioned, Theorem 1.2 deals with the L^∞ norm and no special assumption is needed on the velocity field v. On the other hand for more general L^p-norms with $1 \le p < \infty$, the divergence-free condition on the vector field v has to be assumed, as one needs to calculate the time derivative of the L^p norm and perform integration by parts.

Theorem 1.4 (Generalized maximum principle, L^p case)**.** *Let $\nu \ge 0$, $d \ge 1$, $0 \le \gamma \le 2$ and $\beta \ge 0$, $\lambda > 1$. Assume the vector field $v = v(t,x)$ is divergence free, i.e. $\nabla \cdot v = 0$. If $\theta = \theta(t,x)$ is a smooth solution of (1.1), then for any $1 \le p \le \infty$, we have*

$$\|\theta(t)\|_p \le e^{Ct}\|\theta_0\|_p, \quad \forall\, t > 0, \tag{1.4}$$

where $C > 0$ is a constant depending only on $(\nu, d, \gamma, \beta, \lambda)$.

Remark 1.5. Both Theorem 1.2 and Theorem 1.4 hold in the periodic boundary condition case.[1] In the periodic setting, the decay condition (1.3) is no longer needed as the periodic domain is compact.

It is an interesting question whether one can prove the sharp constant is $C = 0$ in both Theorem 1.2 and Theorem 1.4. We conjecture this is indeed the case at least for a generic set of parameters.

The rest of this article is organized as follows. In Section 2 we introduce the nonlocal decomposition for both the operator $\mathcal{A}$ and the operator $\mathcal{A}_1$. In Section 3 we give the proof for Theorem 1.2 and Theorem 1.4 for the operator $\mathcal{A}_1$. The case $0 \le \gamma \le 1$ of $\mathcal{A}$ is also covered there. In Section 4 we complete the proof of the main theorems for the operator $\mathcal{A}$ in the regime $1 < \gamma \le 2$.

We conclude the introduction by setting up some

Notations.

- For any two quantities X and Y, we denote $X \lesssim Y$ if $X \le CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. We shall write $X \lesssim_{Z_1, Z_2, \cdots, Z_k} Y$ if $X \le CY$ and the constant C depends on the quantities $(Z_1, \cdots, Z_k)$. Similarly we define $\gtrsim_{Z_1,\cdots,Z_k}$ and $\sim_{Z_1,\cdots,Z_k}$.
- For any f on $\mathbb{R}^d$, we denote the Fourier transform of f has

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x}\, dx.$$

  The inverse Fourier transform of any g is given by

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} g(\xi) e^{ix\cdot\xi}\, d\xi.$$

[1] We thank Edriss Titi for this suggestion.

- For any real number x, the sign function $\mathrm{sgn}(x)$ is defined as follows
$$\mathrm{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$
For any complex z with $\mathrm{Re}(z) > 0$, the Gamma function $\Gamma(z)$ is given by the expression
$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$
- We will also occasionally need to use the Littlewood-Paley frequency projection operators. Let $\varphi(\xi)$ be a smooth bump function supported in the ball $|\xi| \le 2$ and equal to one on the ball $|\xi| \le 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$ we define the Littlewood-Paley operators
$$\begin{aligned} \widehat{P_{\le N} f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi), \\ \widehat{P_{> N} f}(\xi) &:= [1-\varphi(\xi/N)]\hat{f}(\xi), \\ \widehat{P_N f}(\xi) &:= [\varphi(\xi/N) - \varphi(2\xi/N)]\hat{f}(\xi). \end{aligned}$$
Similarly we can define $P_{<N}$, $P_{\ge N}$, and $P_{M<\cdot\le N} := P_{\le N} - P_{\le M}$, whenever M and N are dyadic numbers.

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2. The nonlocal decomposition

We start with the following lemma which establishes the nonlocal decomposition of the log-modulated fractional dissipation operator $\frac{|\nabla|^\gamma}{\log^\beta(\lambda+|\nabla|)}$ in the regime $0 \le \gamma \le 1$. One should notice the subtle difference between this operator and the operator $\mathcal{A}_1$ in the logarithmic term. By a simple change of variable $|\xi| \to |\xi|^2$, the decomposition of the operator $\mathcal{A}_1$ is addressed in the next corollary. After that we establish the decomposition for the operator $\mathcal{A}$ in the regime $1 < \gamma \le 2$. The proof will be more involved due to certain first order negative corrections.

Lemma 2.1 (Nonlocal decomposition, case $0 \le \gamma \le 1$)**.** *Let $d \ge 1$, $0 \le \gamma \le 1$ and $\beta > 0$, $\lambda > 1$. Then we have the decomposition:*
$$\frac{|\nabla|^\gamma}{\log^\beta(\lambda+|\nabla|)} = C_\beta \int_0^\gamma \tau^{\beta-1} |\nabla|^{\gamma-\tau}\, d\tau + P, \tag{2.1}$$
where P is a smooth Fourier multiplier which maps L^p to L^p for all $1 \le p \le +\infty$. More precisely, for any function f
$$(Pf)(x) = (K * f)(x) = \int K(x-y) f(y)\, dy,$$
and $\|K\|_{L^1_x} \le Const.$

Proof of Lemma 2.1. On the Fourier side the identity (2.1) is equivalent to the following
$$\frac{|\xi|^\gamma}{\log^\beta(\lambda+|\xi|)} = C_\beta \int_0^\gamma \tau^{\beta-1} |\xi|^{\gamma-\tau}\, d\tau + P(\xi). \tag{2.2}$$

To show (2.2) we start with the simple identity

$$\frac{1}{\log^{\beta}(\lambda+|\xi|)} = \frac{1}{\Gamma(\beta)}\int_0^{\infty} \tau^{\beta-1}(\lambda+|\xi|)^{-\tau}\,d\tau. \tag{2.3}$$

Hence

$$\begin{aligned}\frac{(\lambda+|\xi|)^{\gamma}}{\log^{\beta}(\lambda+|\xi|)} =& \frac{1}{\Gamma(\beta)}\int_0^{\gamma} \tau^{\beta-1}(\lambda+|\xi|)^{\gamma-\tau}\,d\tau \\ &+ \frac{1}{\Gamma(\beta)}\int_{\gamma}^{\infty} \tau^{\beta-1}(\lambda+|\xi|)^{\gamma-\tau}\,d\tau.\end{aligned}$$

We then set $C_\beta = \frac{1}{\Gamma(\beta)}$ and obtain (2.2) with

$$P(\xi) = C_\beta \int_0^{\gamma} \tau^{\beta-1}\Big((\lambda+|\xi|)^{\gamma-\tau} - |\xi|^{\gamma-\tau}\Big)\,d\tau \tag{2.4}$$

$$+ C_\beta \int_{\gamma}^{\infty} \tau^{\beta-1}(\lambda+|\xi|)^{\gamma-\tau}\,d\tau \tag{2.5}$$

$$+ \frac{|\xi|^{\gamma} - (\lambda+|\xi|)^{\gamma}}{\log^{\beta}(\lambda+|\xi|)}. \tag{2.6}$$

It remains for us to show the L^1 boundedness of $\mathcal{F}^{-1}(P)$. We first deal with the piece (2.4). By the Fundamental Theorem of Calculus, we have for any $0 \le t \le 1$,

$$(\lambda+|\xi|)^t - |\xi|^t = \int_0^{\lambda} t(s+|\xi|)^{t-1}\,ds. \tag{2.7}$$

If $0 < t < 1$, then

$$(s+|\xi|)^{t-1} = \frac{1}{\Gamma(1-t)}\int_0^{\infty} y^{-t}e^{-y(s+|\xi|)}\,dy.$$

Since for $y > 0$ the Poisson kernel $\mathcal{F}^{-1}(e^{-y|\xi|})$ is positive, it follows easily that $\mathcal{F}^{-1}((s+|\xi|)^{t-1})$ is a non-negative function and furthermore,

$$\|\mathcal{F}^{-1}((s+|\xi|)^{t-1})\|_{L^1_x} = s^{t-1}.$$

By (2.7), we get

$$\|\mathcal{F}^{-1}((\lambda+|\xi|)^t - |\xi|^t)\|_{L^1_x} = \lambda^t. \tag{2.8}$$

Plugging the above estimate into (2.4), we get

$$\|\mathcal{F}^{-1}(2.4)\|_{L^1_x} \le C_\beta \int_0^{\gamma} \tau^{\beta-1}\lambda^{\gamma-\tau}\,d\tau < \infty,$$

which is clearly good for us.

For (2.5), we just note that for $\tau > \gamma$,

$$\|\mathcal{F}^{-1}((\lambda+|\xi|)^{\gamma-\tau})\|_{L^1_x} = \lambda^{\gamma-\tau},$$

and hence

$$\|\mathcal{F}^{-1}(2.5)\|_{L^1_x} \le C_\beta \int_{\gamma}^{\infty} \tau^{\beta-1}\lambda^{\gamma-\tau}\,d\tau < +\infty,$$

where we used the fact that $\lambda > 1$.

Finally we deal with the contribution of (2.6). By (2.3), it is obvious that

$$\Big\|\mathcal{F}^{-1}\big(\frac{1}{\log^\beta(\lambda+|\xi|)}\big)\Big\|_{L^1_x}=\frac{1}{\log^\beta\lambda}<\infty,$$

since $\lambda>1$. Now in (2.6) we may assume $0<\gamma<1$ (the cases $\gamma=0$ and $\gamma=1$ are trivial). By (2.8) and Young's inequality, we get

$$\begin{aligned}\|\mathcal{F}^{-1}(2.6)\|_{L^1_x}&\le\|\mathcal{F}^{-1}\big(\frac{1}{\log^\beta(\lambda+|\xi|)}\big)\|_{L^1_x}\cdot\|\mathcal{F}^{-1}((\lambda+|\xi|)^\gamma-|\xi|^\gamma)\|_{L^1_x}\\&\le\frac{1}{\log^\beta\lambda}\cdot\lambda^\gamma<\infty.\end{aligned}$$

□

By a simple substitution $|\xi|\to|\xi|^2$, we can deduce the nonlocal decomposition of the operator $\mathcal{A}_1$ from Lemma 2.1. Of course, one still needs to check the L^1 boundedness of the error term under such nonlinear substitution.

Corollary 2.2 (Nonlocal decomposition for the operator $\mathcal{A}_1$)**.** *Let $d\ge1$, $0\le\gamma\le2$ and $\beta>0$, $\lambda>1$. Then we have the decomposition:*

$$\frac{|\nabla|^\gamma}{\log^\beta(\lambda-\Delta)}=C_\beta\int_0^\gamma\tau^{\beta-1}|\nabla|^{\gamma-\tau}\,d\tau+P,\tag{2.9}$$

where P is a smooth Fourier multiplier which maps L^p to L^p for all $1\le p\le+\infty$. More precisely, for any function f

$$(Pf)(x)=(K*f)(x)=\int_{\mathbb{R}^d}K(x-y)f(y)\,dy,$$

and $\|K\|_{L^1_x}\le Const.$

Proof of Corollary 2.2. On the Fourier side, the identity (2.9) is equivalent to the following (the value of the constant C_β can be adjusted slightly)

$$\frac{|\xi|^\gamma}{\log^\beta(\lambda+|\xi|^2)}=C_\beta\int_0^{\frac{\gamma}{2}}\tau^{\beta-1}(|\xi|^2)^{\frac{\gamma}{2}-\tau}\,d\tau+P(\xi).\tag{2.10}$$

By using a similar derivation as in the beginning part of the proof of Lemma 2.1 (see in particular (2.3)–(2.6), and replace $|\xi|$ by $|\xi|^2$, γ by $\gamma/2$), we obtain (2.10) with

$$\begin{aligned}P(\xi)=C_\beta&\int_0^{\frac{\gamma}{2}}\tau^{\beta-1}\Big((\lambda+|\xi|^2)^{\frac{\gamma}{2}-\tau}-(|\xi|^2)^{\frac{\gamma}{2}-\tau}\Big)\,d\tau\\&+C_\beta\int_{\frac{\gamma}{2}}^{\infty}\tau^{\beta-1}(\lambda+|\xi|^2)^{\frac{\gamma}{2}-\tau}\,d\tau\\&+\frac{(|\xi|^2)^{\frac{\gamma}{2}}-(\lambda+|\xi|^2)^{\frac{\gamma}{2}}}{\log^\beta(\lambda+|\xi|^2)}.\end{aligned}$$

Note that $0\le\frac{\gamma}{2}\le1$ and the fact $\|\mathcal{F}^{-1}(a+|\xi|^2)^{-s}\|_{L^1_x}=a^{-s}$ for any $s>0$ and $a>0$. By using a similar analysis as in the proof of Lemma 2.1, it is then not difficult to check that $\mathcal{F}^{-1}(P)$ is an L^1 bounded kernel. □

We now consider the more involved $1<\gamma\le2$ case for the operator $\mathcal{A}$. One should compare the decomposition (2.11) with (2.1).

Lemma 2.3 (Nonlocal decomposition, case $1<\gamma\le 2$)**.** *Let $d\ge 1$, $1<\gamma\le 2$ and $\beta>0$, $\lambda>1$. Then we have the decomposition:*

$$\begin{aligned}&\frac{|\nabla|^\gamma}{\log^\beta(\lambda+|\nabla|)}\\ &=C_\beta\int_{\gamma-1}^{\gamma}\tau^{\beta-1}|\nabla|^{\gamma-\tau}d\tau+C_\beta\int_0^{\gamma-1}\tau^{\beta-1}\Big(|\nabla|^{\gamma-\tau}-\lambda\,\tau|\nabla|^{\gamma-\tau-1}\Big)\,d\tau+P,\end{aligned}\tag{2.11}$$

where P is a smooth Fourier multiplier which maps L^p to L^p for all $1\le p\le+\infty$. More precisely, for any function f

$$(Pf)(x)=(K*f)(x)=\int K(x-y)f(y)\,dy,$$

and $\|K\|_{L^1_x}\le Const.$

Proof of Lemma 2.3. Throughout this proof we shall use the letter P to denote the symbol of an $L^1\to L^1$ bounded operator. For the convenience of notation, we allow the value of P to vary from line to line. We begin with two elementary estimates. Let $\phi\in C_c^\infty(\mathbb{R}^d)$ be a radial smooth cut-off function such that $\phi(x)=1$ for $|x|\le 2$ and $\phi(x)=0$ for $|x|\ge 3$. For any constant $C>0$ define $\phi_{<C}(x):=\phi(x/C)$ and $\phi_{>C}(x)=1-\phi_{<C}(x)$. Then for any $\gamma>0$, $C>0$, $s\ge 0$, we have the following

$$\|\mathcal{F}^{-1}\Big(\big(\frac{|\xi|}{\lambda+|\xi|}\big)^\gamma\Big)\|_{L^1_x\to L^1_x}<\infty,\tag{2.12}$$

$$\|\mathcal{F}^{-1}\Big(|\xi|^{-s}\phi_{>C}(\xi)\Big)\|_{L^1_x}\le C_1(1+s)^{d+1}C^{-s},\tag{2.13}$$

where the constant C_1 is independent of s. To prove (2.12), one can use a scaling argument to reduce to the $\lambda=1$ case. The result then follows easily from the binomial expansion of $(|\xi|/(1+|\xi|))^\gamma=(1-\frac{1}{1+|\xi|})^\gamma=\sum_{n\ge 0}C_{\gamma,n}(1+|\xi|)^{-n}$, the L^1-boundedness of the operators $(1+|\xi|)^{-n}$ (namely $\|\mathcal{F}^{-1}((1+|\xi|)^{-n})\|_{L^1_x\to L^1_x}\le 1$ for any $n\ge 0$), and the fact that $\sum_{n\ge 0}|C_{\gamma,n}|<+\infty$ (note that $C_{n,\gamma}$ has a definite sign for n sufficiently large). To prove (2.13) one can again use scaling to reduce to the case $C=1$. For $s\ge 2$ the result is obvious by using integration by parts. For $0<s\le 2$, we note that

$$\|\mathcal{F}^{-1}\Big((1+|\xi|)^{-s}\phi_{>C}(\xi)\Big)\|_{L^1_x}\lesssim 1,$$

and on the support of $\phi_{>C}(\xi)$,

$$|\xi|^{-s}-(1+|\xi|)^{-s}\sim|\xi|^{-s-1}.$$

One can then use the Littlewood-Paley operators to bound (since we are summing over $N\ge N_0$ the convergence in L^1 is of no problem):

$$\begin{aligned}\|\mathcal{F}^{-1}(|\xi|^{-s}\phi_{>1})\|_{L^1_x\to L^1_x}&\lesssim 1+\|P_{\gtrsim 1}((|\nabla|^{-s}-(1+|\nabla|)^{-s})\delta_0)\|_{L^1_x}\\ &\lesssim 1+\sum_{\text{N dyadic}:N\gtrsim 1}N^{-s-1}\|P_N\delta_0\|_{L^1_x}\\ &\le Const.\end{aligned}$$

Observe that the implied constants are uniform in s since $0<s\le 2$. This settles (2.13).

By (2.3), we write

$$\Gamma(\beta)\frac{|\xi|^\gamma}{\log^\beta(\lambda+|\xi|)} = \int_0^{\gamma-1} \tau^{\beta-1}|\xi|^\gamma(\lambda+|\xi|)^{-\tau}\,d\tau \tag{2.14}$$

$$+\int_{\gamma-1}^{\gamma} \tau^{\beta-1}|\xi|^\gamma(\lambda+|\xi|)^{-\tau}\,d\tau \tag{2.15}$$

$$+\int_{\gamma}^{\infty} \tau^{\beta-1}|\xi|^\gamma(\lambda+|\xi|)^{-\tau}\,d\tau. \tag{2.16}$$

We first deal with (2.16). Rewrite

$$|\xi|^\gamma(\lambda+|\xi|)^{-\tau} = (\frac{|\xi}{\lambda+|\xi|})^\gamma \cdot (\lambda+|\xi|)^{\gamma-\tau}.$$

By using (2.12) we obtain

$$\|\mathcal{F}^{-1}(2.16)\|_{L^1_x} \lesssim \int_\gamma^\infty \tau^{\beta-1}\lambda^{\gamma-\tau}\,d\tau < \infty.$$

Next we turn to (2.15). By inserting a smooth cut-off function $\phi_{>10\lambda}(\xi)$, We have

$$\begin{aligned}(2.15) &= \int_{\gamma-1}^{\gamma} \tau^{\beta-1}|\xi|^{\gamma-\tau}(1+\frac{\lambda}{|\xi|})^{-\tau}\,d\tau\phi_{>10\lambda}(\xi) + P \\ &= \int_{\gamma-1}^{\gamma} \tau^{\beta-1}|\xi|^{\gamma-\tau}\,d\tau\phi_{>10\lambda}(\xi) + P \\ &\quad + \int_{\gamma-1}^{\gamma} \tau^{\beta-1}|\xi|^{\gamma-\tau}((1+\frac{\lambda}{|\xi|})^{-\tau}-1)\,d\tau\phi_{>10\lambda}(\xi). \qquad (2.17)\end{aligned}$$

On the other hand, by using the binomial expansion of the function $(1+t)^{-s} = \sum_{n\ge 0} C_{n,s}t^n$ and the estimate (2.13), it is not difficult to check that

$$\begin{aligned}\|\mathcal{F}^{-1}(2.17)\|_{L^1_x} &\lesssim \int_{\gamma-1}^{\gamma} \tau^{\beta-1}\Big(\sum_{n\ge1}|C_{n,\tau}|\cdot\lambda^n\cdot(10\lambda)^{-(n+\tau-\gamma)}\cdot(1+n+\tau-\gamma)^{d+1}\Big)\,d\tau \\ &< +\infty.\end{aligned}$$

Hence we have proved

$$(2.15) = \int_{\gamma-1}^{\gamma} \tau^{\beta-1}|\xi|^{\gamma-\tau}\,d\tau + P.$$

We turn now to the final piece (2.14). The main idea is similar to that of (2.15). We again insert a smooth cut-off $\phi_{>10\lambda}(\xi)$ and observe that when $|\xi| \ge 10\lambda$, we have

$$(1+\frac{\lambda}{|\xi|})^{-\tau} = 1-\tau\frac{\lambda}{|\xi|} + \sum_{n\ge2} C_{n,\gamma}(\frac{\lambda}{|\xi|})^n.$$

Note that $0 \le \tau \le \gamma-1$ and we have to keep terms up to the linear term. Then clearly

$$(2.14) = \int_0^{\gamma-1} \tau^{\beta-1}(|\xi|^{\gamma-\tau} - \tau\lambda|\xi|^{\gamma-\tau-1})\,d\tau + P.$$

The desired decomposition (2.11) now follows.

$\square$

3. Proof of Theorem 1.2 and Theorem 1.4 for the operator $\mathcal{A}_1$

In this section we give the proofs of Theorems 1.2 and 1.4 for the operator $\mathcal{A}_1$. The proofs for the operator $\mathcal{A}$ is slightly more involved and will be given in the next section.

Proof of Theorem 1.2 for the operator $\mathcal{A}_1$. Assume first $0 < \gamma \le 2$ and $\beta > 0$. By (2.9) we can write

$$\mathcal{A}_1 = L + P, \tag{3.1}$$

where $L = C_\beta \int_0^\gamma \tau^{\beta-1} |\nabla|^{\gamma-\tau} d\tau$ and $\|P\|_{L_x^1} < +\infty$. Now take $\lambda_1 > \nu \|P\|_{L_x^1}$ and define $f(t,x) = e^{-\lambda_1 t}\theta(t,x)$. Fix $T > 0$ and consider

$$\sup_{0 \le t \le T,\, x \in \mathbb{R}^d} |f(t,x)| = M > 0.$$

Without loss of generality we can assume

$$\sup_{0 \le t \le T,\, x \in \mathbb{R}^d} f(t,x) = M > 0.$$

By using the decay condition (1.2) and a simple compactness argument in t, we conclude that there exists (t_0, x_0) such that

$$f(t_0, x_0) = M.$$

We now show that $t_0 = 0$. Indeed if $0 < t_0 \le T$, we compute

$$\begin{aligned}(\partial_t f)(t_0,x_0) &= -\lambda_1 f(t_0,x_0) - \nu(\mathcal{A}_1 f)(t_0,x_0) \\ &= -\lambda_1 M - \nu(Lf)(t_0,x_0) - \nu(Pf)(t_0,x_0).\end{aligned} \tag{3.2}$$

Now by Corollary 2.2, we have

$$\|Pf(t_0)\|_{L_x^\infty} \le \|P\|_{L_x^1} \|f(t_0)\|_{L_x^\infty} \le \|P\|_{L_x^1} \cdot M. \tag{3.3}$$

For any $0 < s < 2$, by using the fractional representation

$$(|\nabla|^s g)(x) = C_s \lim_{\epsilon \to 0} \int_{|y-x|>\epsilon} \frac{g(x)-g(y)}{|x-y|^{d+s}}\, dy,$$

it is easy to see $(|\nabla|^s f)(t_0,x_0) \ge 0$ and hence

$$(Lf)(t_0,x_0) \ge 0. \tag{3.4}$$

Plugging (3.3) and (3.4) into (3.2), we reach a contradiction:

$$(\partial_t f)(t_0,x_0) < -(\lambda_1 - \nu\|P\|_{L_x^1})M < 0.$$

Therefore we conclude that $t_0 = 0$ and clearly the estimate (1.3) follows.

It remains to prove the case $\gamma = 0$ and $\beta > 0$. But in this case the operator $\log^{-\beta}(\lambda - \Delta)$ corresponds to an L^1-bounded convolution kernel. Hence we just need to repeat the previous argument with $\mathcal{A}_1 = P$ and $\lambda_1 > \nu\|P\|_{L_x^1}$. We omit the repetitive details.

□

Finally we complete the

Proof of Theorem 1.4 for the operator $\mathcal{A}_1$. Without loss of generality we assume $\nu > 0$, $0 < \gamma \le 2$ and $\beta > 0$. Let $1 \le p < \infty$. Multiplying both sides of (1.1) by $|\theta|^{p-1}\mathrm{sgn}(\theta)$, integrating by parts and using the fact that v is divergence free, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}(\|\theta(t)\|_p^p) &= -\nu \int_{\mathbb{R}^d} (\mathcal{A}_1\theta)|\theta|^{p-1}\mathrm{sgn}(\theta)\,dx \\ &= -\nu \int_{\mathbb{R}^d} (L\theta)|\theta|^{p-1}\mathrm{sgn}(\theta)\,dx - \nu\|(P\theta)|\theta|^{p-1}\|_{L_x^1}, \end{aligned} \tag{3.5}$$

where in the last equality we have used the decomposition (3.1).

Since for any $0 \le s < 2$, $1 \le p < \infty$, we have

$$\int_{\mathbb{R}^d} (|\nabla|^s\theta)|\theta|^{p-1}\mathrm{sgn}(\theta)\,dx \ge 0,$$

it follows easily that

$$\int_{\mathbb{R}^d} (L\theta)|\theta|^{p-1}\mathrm{sgn}(\theta)\,dx \ge 0.$$

By Hölder, we have

$$\|(P\theta)|\theta|^{p-1}\|_{L_x^1} \le \|P\|_{L_x^1}\|\theta\|_{L_x^p}^p.$$

Plugging the above estimates into (3.5) and integrating in time, we get for any $1 \le p < \infty$,

$$\|\theta(t)\|_{L_x^p} \le e^{\nu\|P\|_{L_x^1}t}\|\theta_0\|_{L_x^p}.$$

The case $p = \infty$ follows by a limiting argument $p \to \infty$. Clearly (1.4) holds by setting $C = \nu\|P\|_{L_x^1}$. □

4. Proof of the main theorems for the operator $\mathcal{A}$

In this section we describe the proofs of the main theorems for the operator $\mathcal{A}$. We shall only need to consider the case $1 < \gamma \le 2$. Thanks to Lemma 2.1 the case $0 \le \gamma \le 1$ is already covered in the previous section. In the case $1 < \gamma \le 2$ we have to use the decomposition (2.11) in Lemma 2.3. The extra complication is due to the negative term $-\lambda\tau|\nabla|^{\gamma-\tau-1}$ which in principle can cause the maximum principle to fail. The way out of this difficulty is to note that the main term $|\nabla|^{\gamma-\tau}$ is stronger than this negative term by an order of $|\nabla|^{-1}$. The following lemma quantifies this observation. In some sense it gives the maximum principle for "mixed" operators.

Lemma 4.1. *Let $0 < s_1 < s_2 < 2$ and $C_1 > 0$. Consider the operator*

$$L = |\nabla|^{s_2} - C_1|\nabla|^{s_1}.$$

Then for any smooth function g which attains its maximum at some point x_0, we have

$$(Lg)(x_0) \ge -C_1C_d\|g\|_\infty \cdot (2-s_1)\Big(1 + \big(s_2(2-s_2)\big)^{-\frac{s_1}{s_2-s_1}}\Big). \tag{4.1}$$

where $C_d > 0$ is some constant depending only on the dimension d. In particular if $s_1 = s_2 - 1$ and $1 < s_2 < 2$, then we have the estimate

$$(Lg)(x_0) \ge -C_1C_d'\|g\|_\infty(1 + (2-s_2)^{1-s_2}), \tag{4.2}$$

where $C_d' > 0$ is another constant depending only on the dimension d.

Proof of Lemma 4.1. To begin we need to derive the explicit constant appearing in the integral representation of the fractional operators $|\nabla|^s$ with $0 < s < 2$. Recall that for any $0 < \alpha < d$, the Riesz potential $|\nabla|^{-\alpha}$ has the following explicit representation (cf. pp 117 of Stein [8])

$$(|\nabla|^{-\alpha} f) = c_{\alpha,d} \int_{\mathbb{R}^d} |x-y|^{-d+\alpha} f(y)\, dy,$$

with

$$c_{\alpha,d} = \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2}) 2^{\alpha} \pi^{\frac{d}{2}}}.$$

For $0 < s < 2$ by writing $|\nabla|^s = -\Delta |\nabla|^{-(2-s)}$ and integrating by parts, we get

$$(|\nabla|^s f) = C_{s,d} \lim_{\epsilon \to 0} \int_{|y-x|>\epsilon} \frac{f(x)-f(y)}{|x-y|^{d+s}}\, dy,$$

where

$$C_{s,d} = s \frac{\Gamma(\frac{d+s}{2})}{2^{2-s} \Gamma(\frac{2-s}{2}) \pi^{\frac{d}{2}}}.$$

By using the asymptotics $\Gamma(z) \sim z^{-1}$ for $z \sim 0$, it is easy to see that

$$C_{s,d} \sim_d \frac{s}{\Gamma(\frac{2-s}{2})} \sim_d s(2-s). \tag{4.3}$$

and in particular for all $0 < s < 2$,

$$C_{s,d} \lesssim_d 1, \tag{4.4}$$

We now write

$$\begin{aligned}\Big((|\nabla|^{s_2} - C_1 |\nabla|^{s_1}) g\Big)(x_0) = {} & C_{s_2,d} \lim_{\epsilon \to 0} \int_{|y-x_0|>\epsilon} \frac{g(x_0)-g(y)}{|x_0-y|^{d+s_2}}\, dy \\ & - C_1 C_{s_1,d} \lim_{\epsilon \to 0} \int_{|y-x_0|>\epsilon} \frac{g(x_0)-g(y)}{|x_0-y|^{d+s_1}}\, dy. \end{aligned} \tag{4.5}$$

Observe that $g(x_0) - g(y) \geq 0$ for all y. We now separate the y-integral into two regimes. The first regime is $\{y : |y - x_0| \leq \min\{1, C_2\}\}$, where $C_2 > 0$ is a constant such that (here we use (4.4) to bound $C_{s_1,d}$)

$$(C_2)^{s_1 - s_2} C_{s_2,d} \gtrsim_d C_1.$$

By using (4.3), we have (the notation $\sim_{C_1,d}$ means up a constant depending on C_1 and d)

$$C_2 \sim_{C_1,d} \Big(s_2(2-s_2)\Big)^{\frac{1}{s_2-s_1}}.$$

In the first regime, it is easy to check that the first integral bounds the second integral in (4.5). The second regime is just the complement $\{y : |y - x_0| > \min\{1, C_2\}\}$. In this case we simply discard the first integral and bound the second

integral by $\|g\|_{L^\infty_x}$ which produces a term of the form (below C_d denotes a constant depending only on the dimension d):

$$\begin{aligned} & - C_1 C_d \frac{s_1}{\Gamma(\frac{2-s_1}{2})} \|g\|_\infty \int_{r>\min\{1,C_2\}} r^{-1-s_1}\, dr \\ \gtrsim_{C_1,d} \quad & - \|g\|_\infty \cdot (2-s_1)\Big(1 + \big(s_2(2-s_2)\big)^{-\frac{s_1}{s_2-s_1}}\Big). \end{aligned}$$

This settles (4.1).

Finally (4.2) is a simple consequence of (4.1). □

The following corollary will be used in the proof of Theorem 1.2.

Corollary 4.2. *Let $d \ge 1$, $1 < \gamma \le 2$ and $\beta > 0$, $\lambda > 1$. Then for any smooth function g which attains its maximum at some point x_0, we have*

$$\Big(\frac{|\nabla|^\gamma}{\log^\beta(\lambda + |\nabla|)} g\Big)(x_0) \ge -(C_{d,\beta,\gamma}\lambda + \|P\|_{L^1_x})\|g\|_\infty,$$

where $C_{d,\beta,\gamma}$ is some constant depending only on (d,α,β,γ), and P is the operator defined in (2.11).

Proof of Corollary 4.2. By Lemma 2.3, Lemma 4.1, and the fact that $(|\nabla|^{\gamma-\tau} g)(x_0) \ge 0$, we have

$$\begin{aligned} & \Big(\frac{|\nabla|^\gamma}{\log^\beta(\lambda + |\nabla|)} g\Big)(x_0) \\ \ge \quad & C_\beta \int_0^{\gamma-1} \tau^{\beta-1} \Big((|\nabla|^{\gamma-\tau} - \lambda\tau |\nabla|^{\gamma-\tau-1}) g\Big)(x_0) d\tau - \|P\|_{L^1_x} \|g\|_\infty \\ \ge \quad & - C_{\beta,d} \cdot \lambda \|g\|_\infty \int_0^{\gamma-1} \tau^\beta (1 + (2-\gamma+\tau)^{1-\gamma+\tau}) d\tau - \|P\|_{L^1_x}\|g\|_\infty \\ \ge & - (C_{d,\beta,\gamma}\lambda + \|P\|_{L^1_x})\|g\|_\infty. \end{aligned}$$

□

We are now ready to complete the

Proof of Theorem 1.2 for the operator $\mathcal{A}$, case $1 < \gamma \le 2$. With the help of Corollary 4.2, the proof is similar to the proof for the operator $\mathcal{A}_1$ in section 3, one only needs to consider $f(t,x) = e^{-\lambda_1 t}\theta(t,x)$ with $\lambda_1 > \nu(C_{d,\beta,\gamma}\lambda + \|P\|_{L^1_x})$, where the constant $C_{d,\beta,\gamma}$ is the same as in Corollary 4.2. The rest of the proof is now the same as in Section 3. We omit the details. □

Next we turn to the proof of Theorem 1.4. The following lemma establishes a form of maximum principle for the mixed operator $L = |\nabla|^s - C_1|\nabla|^{s-1}$ in the L^p, $1 \le p < \infty$ setting.

Lemma 4.3. *Let $1 < \gamma \le 2$, $1 < s < \gamma$, $C_1 > 0$ and consider the operator*

$$L = |\nabla|^s - C_1(\gamma - s)|\nabla|^{s-1}.$$

Then for any $1 \le p < \infty$ and any smooth $g \in L^p$, we have the bound

$$\int_{\mathbb{R}^d} (Lg)|g|^{p-1} sgn(g)\, dx \ge -C_1 C_{d,\gamma} \|g\|_p^p, \tag{4.6}$$

where $C_{d,\gamma} > 0$ is some constant depending only on (d,γ).

Proof of Lemma 4.3. The proof is analogous to that of Corollary 4.2 with suitable modifications. Without loss of generality we assume $C_1 = 1$. We begin with a simple tail estimate. For any $A > 0$, we have

$$
\begin{aligned}
&(s-1)\int_{\mathbb{R}^d}\int_{|y-x|>A}\frac{|f(x)-f(y)|}{|x-y|^{d+s-1}}(|f(x)|^{p-1}+|f(y)|^{p-1})\,dx\,dy\\
\le&C_d\|f\|_p^p\cdot A^{1-s}, \qquad (4.7)
\end{aligned}
$$

where $C_d > 0$ is a constant depending only on the dimension d. The estimate (4.7) is a simple consequence of the Young's inequality

$$
|f(y)|\cdot|f(x)|^{p-1}\le\frac{p-1}{p}|f(x)|^p+\frac{1}{p}|f(y)|^p,
$$

and Fubini.

Next we need to transform the LHS of (4.6) suitably. By a symmetrization in the variable x and y, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d}(|\nabla|^s g)|g|^{p-1}\mathrm{sgn}(g)\,dx\\
=&C_{s,d}\int\int\frac{g(x)-g(y)}{|x-y|^{d+s}}dy|g(x)|^{p-1}\mathrm{sgn}(g(x))\,dx\\
=&\frac{C_{s,d}}{2}\int\int\frac{h(x,y)}{|x-y|^{d+s}}\,dx\,dy, \qquad (4.8)
\end{aligned}
$$

where

$$
h(x,y)=(g(x)-g(y))\Big(|g(x)|^{p-1}\mathrm{sgn}(g(x))-|g(y)|^{p-1}\mathrm{sgn}(g(y))\Big).
$$

Note that for any real numbers a, b, we have

$$
(a-b)\Big(|a|^{p-1}\mathrm{sgn}(a)-|b|^{p-1}\mathrm{sgn}(b)\Big)\ge 0.
$$

Therefore $h(x, y) \ge 0$ for all x, y. The advantage of the expression (4.8) is that the integrand is always non-negative. Similar expression holds for the operator $|\nabla|^{s-1}$.

Let $A > 0$ be a constant whose value will be specified later. By using (4.8) and (4.3), we have

$$
\begin{aligned}
\text{LHS of (4.6)}\ge{}& C_d s(2-s)\int_{\mathbb{R}^d}\int_{|y-x|<A}\frac{h(x,y)}{|x-y|^{d+s}}\,dx\,dy && (4.9)\\
&-C_d'(s-1)(\gamma-s)\int_{\mathbb{R}^d}\int_{|y-x|<A}\frac{h(x,y)}{|x-y|^{d+s-1}}\,dx\,dy && (4.10)\\
&-C_d'(s-1)(\gamma-s)\int_{\mathbb{R}^d}\int_{|y-x|>A}\frac{h(x,y)}{|x-y|^{d+s-1}}\,dx\,dy, && (4.11)
\end{aligned}
$$

where C_d and C_d' are constants depending only on the dimension d.

By (4.7), we have

$$
(4.11)\ge -C_d'\|g\|_p^pA^{1-s}(\gamma-s). \qquad (4.12)
$$

On the other hand, by choosing $A = C_d/C_d'$, we have

$$
C_d\cdot s(2-s)\cdot\frac{1}{A}\ge C_d'(s-1)(\gamma-s),
$$

and

$$(4.9) + (4.10) \geq 0.$$

Substituting the value of A into (4.12), we obtain (4.6).

□

Finally we are ready to complete the

Proof of Theorem 1.4 for the operator $\mathcal{A}$ in the regime $1 < \gamma \leq 2$. Thanks to Lemma 2.3 and Lemma 4.3, we essentially only have to repeat the proof for the operator $\mathcal{A}_1$ in Section 3. In place of (3.5), we have

$$\frac{1}{p}\frac{d}{dt}(\|\theta(t)\|_p^p) \leq \nu(\|P\|_{L^1_x} + \lambda C_{d,\gamma,\beta})\|\theta(t)\|_p^p.$$

The estimate (1.4) follows immediately.

□

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(H. Dong) Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA

E-mail address: Hongjie_Dong@brown.edu

(D. Li) Department of Mathematics, University of British Columbia, Vancouver BC Canada V6T 1Z2

E-mail address: mpdongli@gmail.com